A NEW NOTION OF PATH LENGTH IN THE PLANE

L. C. HOEHN, L. G. OVERSTEEGEN, AND E. D. TYMCHATYN

Abstract. A path in the plane is a continuous function γ from the unit interval into the plane. The Euclidean length of a path in the plane, when defined, has the following properties: it is invariant under isometries of the plane, it is monotone with respect to subpaths, and for any two points in the plane the path mapping homeomorphically onto the straight line segment joining them is the unique, nowhere constant path (up to homeomorphism) of minimal length connecting them. However, the Euclidean length does not behave well with respect to limits – even when a sequence of paths γ_i converges uniformly to a path γ it is not necessarily true that their lengths converge. Moreover, for many paths the Euclidean length is not defined (i.e. not finite).

In this paper we introduce an alternative notion of length of a path, len, which has the above three properties, and is such that the length of any path is defined and finite. This enables one to compare the efficiency (shortness) of two paths between a given pair of points, even if their Euclidean lengths are infinite. In addition, if a sequence of paths γ_i converges uniformly to a path γ, them $\lim \mathsf{len}(\gamma_i) = \mathsf{len}(\gamma)$. Moreover, this notion of length is defined for any map γ from a locally connected continuum into the plane.

We apply this notion of length to obtain a characterization of those families of paths which can be reparameterized to be equicontinuous. We also prove the existence of a unique shortest path in the closure of a given homotopy class of paths in an arbitrary plane domain, which extends known results for simply connected domains.

1. Introduction

A *path* is a continuous function γ from a closed interval $[a,b] \subset \mathbb{R}$ to the plane $\mathbb{C}$. Given $z_1, z_2 \in \mathbb{C}$, denote by $\overline{z_1 z_2}$ the straight line segment path $t \mapsto (1-t)z_1 + tz_2$, $t \in [0,1]$.

Given a path $\gamma : [a,b] \to \mathbb{C}$, the *Euclidean path length* of γ, denoted $L_E(\gamma)$, is defined by the formula

$$L_E(\gamma) = \sup \left\{ \sum_{i=1}^{n} d(\gamma(x_{i-1}), \gamma(x_i)) : a = x_0 < x_1 < \cdots < x_n = b,\ n \in \mathbb{Z}^+ \right\} \in [0, \infty],$$

where d denotes the Euclidean distance.

The function $L = L_E$ satisfies the following basic properties for the path $\gamma : [a,b] \to \mathbb{C}$:

(L1) If $A \subset [a,b]$ is a closed subinterval, then $L(\gamma{\restriction}_A) \leq L(\gamma)$;
(L2) If $c \in (a,b)$, then $L(\gamma) = L(\gamma{\restriction}_{[a,c]}) + L(\gamma{\restriction}_{[c,b]})$;

Date: August 3, 2011.
2010 *Mathematics Subject Classification.* Primary 54F15; Secondary 54F35, 54F50.
Key words and phrases. path, length, plane, locally connected continuum.
The second named author was partially supported by NSF grant DMS-0906316.
The third named author was partially supported by NSERC grant OGP-0005616.

(L3) If $\Phi : \mathbb{C} \to \mathbb{C}$ is an isometry, then $L(\Phi \circ \gamma) = L(\gamma)$;

(L4) $L(\overline{\gamma(a)\gamma(b)}) \leq L(\gamma) = \sup \left\{ \sum_{i=1}^{n} L(\overline{\gamma(x_{i-1})\gamma(x_i)}) : a = x_0 < x_1 < \cdots < x_n = b,\ n \in \mathbb{Z}^+ \right\}$;

and moreover, we have

(L5) $L(\overline{01}) = 1$.

Conversely, any function L defined on the set of all paths which satisfies the properties **(L1)** through **(L5)** must be equal to L_E (and any function L which satisfies the properties **(L1)** through **(L4)** must be a scalar multiple $c \cdot L_E$ of L_E, where $c = L(\overline{01})$). Indeed, one can use properties **(L1)**, **(L2)**, **(L3)**, and **(L5)** to show that $L(\overline{ab}) = b - a$ for any $a, b \in \mathbb{R} \subset \mathbb{C}$ with $a < b$. Then by property **(L3)** it follows that $L(\overline{z_1 z_2}) = L_E(\overline{z_1 z_2}) = d(z_1, z_2)$ for any $z_1, z_2 \in \mathbb{C}$. We then conclude by **(L4)** that $L(\gamma) = L_E(\gamma)$ for all paths γ.

There are a number of results in metric geometry pertaining to when a given metric on a Euclidean space is equal to the Euclidean metric; [1], [3], and [8] each survey a variety of such results. Much of this work is related to Hilbert's fourth problem. The length function introduced in this paper contributes to the corresponding program for path length functions by illustrating that there are other length functions which have many properties in common with the Euclidean length.

In light of the above discussion, to provide a genuinely different path length function from the Euclidean length, one must give up at least one of the properties **(L1)** through **(L4)**. In Section 2, we define a path length function, called "len", such that $L = \mathsf{len}$ satisfies properties **(L1)** and **(L3)** (see Propositions 4(iii) and 5(i) below), as well as the following weaker forms of **(L2)** and **(L4)** (see Propositions 5(ii) and 6):

(L2′) If $c \in (a, b)$, then $L(\gamma) \leq L(\gamma\restriction_{[a,c]}) + L(\gamma\restriction_{[c,b]})$;

(L4′) $L(\overline{\gamma(a)\gamma(b)}) \leq L(\gamma) \leq \sup \left\{ \sum_{i=1}^{n} L(\overline{\gamma(x_{i-1})\gamma(x_i)}) : a = x_0 < x_1 < \cdots < x_n = b,\ n \in \mathbb{Z}^+ \right\}$.

Furthermore, this length function has the following additional properties not enjoyed by the Euclidean length L_E:

- len is defined for any continuous function γ from a locally connected continuum X to $\mathbb{C}$;
- $\mathsf{len}(\gamma) < 1$ for any such γ;
- If $\langle \gamma_m \rangle_{m=0}^{\infty}$ is a sequence of maps $X \to \mathbb{C}$ converging uniformly to $\gamma : X \to \mathbb{C}$, then $\mathsf{len}(\gamma_m) \to \mathsf{len}(\gamma)$ as $m \to \infty$ (see Proposition 7). In particular, it follows that $\mathsf{len}(\gamma_m) \to 0$ if and only if $\mathrm{diam}(\gamma_m(X)) \to 0$.

Moreover, this length function can differentiate between paths whose Euclidean lengths are infinite. For instance, if $\gamma : [a, b] \to \mathbb{C}$ is a path and $[c, d]$ is a subinterval of $[a, b]$ such that γ is non-constant on at least one component of $[a, b] \setminus [c, d]$, then $\mathsf{len}(\gamma) > \mathsf{len}(\gamma\restriction_{[c,d]})$, even if both of these paths have infinite Euclidean length.

A very similar function is developed by Cannon et. al. in [4], which is called the *total oscillation* of a path; however, the total oscillation is not invariant under isometries of the plane.

After establishing the above properties in Section 3, we use len in Section 4 to obtain a standard parameterization of a path in the plane. This yields a characterization of those families of paths in the plane which may be reparameterized so as to be equicontinuous. This result extends classical results on equicontinuous families of paths having finite Euclidean length.

In Section 5, we turn our attention to paths in a plane domain Ω. Using the results of Section 4, we prove that there exists a unique shortest path in the closure of any given homotopy class of paths in Ω. This extends work of Bourgin and Renz [2], who established this result for simply connected domains with simple closed curve boundary.

2. Definition of the function len

For $j \in \mathbb{Z}$, let S_j denote the closed horizontal strip $\{a + ib : a \in \mathbb{R}, b \in [j, j+1]\}$ in the plane $\mathbb{C}$. Given $x, t \in [0,1]$, $\mu \in (0,1]$, and $j \in \mathbb{Z}$, let $S_j^{x,t,\mu} = \mu e^{t\pi i}(S_j + ix)$. If $A \subset \mathbb{C}$, define $\|A\|_t = \operatorname{diam}(\operatorname{proj}_t^\perp(A))$, where $\operatorname{proj}_t^\perp$ denotes the orthogonal projection of $\mathbb{C}$ onto the line $\{re^{(t+\frac{1}{2})\pi i} : r \in \mathbb{R}\}$ and diam denotes the diameter in the Euclidean metric.

A *generalized path* is a continuous function γ from a locally connected continuum X to $\mathbb{C}$. Fix a generalized path $\gamma : X \to \mathbb{C}$.

The following lemma will be used in the definition of the function len below.

Lemma 1. *For any $x, t, \mu \in [0,1] \times [0,1] \times (0,1]$ and any $\varepsilon > 0$, there are only finitely many components C of the sets $\gamma^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) with $\|\gamma(C)\|_t \geq \varepsilon$.*

Proof. We may assume that $\varepsilon \leq \frac{\mu}{2}$. Suppose for a contradiction that there are infinitely many distinct components $\{C_n\}_{n=0}^\infty$ of the sets $\gamma^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) with $\|\gamma(C)\|_t \geq \varepsilon$.

For each n let $j(n) \in \mathbb{Z}$ be the integer for which $\gamma(C_n) \subset S_{j(n)}^{x,t,\mu}$, and let $p_n \in C_n$ be such that $d(\gamma(p_n), \partial S_{j(n)}^{x,t,\mu}) = \varepsilon$. Observe that by local connectivity of X, for each n we have $\gamma(\partial C_n) \subset \partial S_{j(n)}^{x,t,\mu}$.

Let $p \in X$ be an accumulation point of the set $\{p_n\}_{n=0}^\infty$, and let U be an open neighborhood of p which is small enough so that $\operatorname{diam}(\gamma(U)) < \varepsilon$. Then for any n such that $p_n \in U$, we have $\gamma(U) \cap \partial S_{j(n)}^{x,t,\mu} = \emptyset$, hence $U \cap \partial C_n = \emptyset$, and so $U \cap C_n$ is closed and open in U. It follows that U cannot be connected, which is a contradiction since X is locally connected. $\square$

Given $x, t \in [0,1]$ and $\mu \in (0,1]$, let $\langle C_n^{x,t,\mu} \rangle_{n=0}^\infty$ enumerate the collection of all components of the sets $\gamma^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\operatorname{proj}_t^\perp$, ordered so that $\|\gamma(C_n^{x,t,\mu})\|_t \geq \|\gamma(C_{n+1}^{x,t,\mu})\|_t$ for all n (this is possible by Lemma 1).

Define

$$L^{x,t,\mu}(\gamma) = \sum_{n=0}^{\infty} \frac{\|\gamma(C_n^{x,t,\mu})\|_t}{2^n}$$

and define the *length of* γ by

$$\mathsf{len}(\gamma) = \int_0^1 \int_0^1 \int_0^1 L^{x,t,\mu}(\gamma)\, dx\, dt\, d\mu.$$

If $X \subset \mathbb{C}$ is a locally connected continuum, define $\mathsf{len}(X) = \mathsf{len}(\mathrm{id}_X)$.

Observe that if σ is any injective function of the non-negative integers to themselves, then

$$(*) \qquad \sum_{n=0}^{\infty} \frac{\|\gamma(C_{\sigma(n)}^{x,t,\mu})\|_t}{2^n} \leq L^{x,t,\mu}(\gamma).$$

It remains to show that the function $L^{x,t,\mu}(\gamma)$ is in fact integrable, so that the above definition of the function len makes sense. This is accomplished in Lemma 3 below.

Lemma 2. *Let C be a component of $\gamma^{-1}(S_j^{x,t,\mu})$ for some x, t, μ, j which has non-degenerate image under the map $\mathrm{proj}_t^\perp$, and let $\varepsilon > 0$. Then there exists a subcontinuum $D \subset C$ such that $\gamma(D) \subset \mathrm{int}(S_j^{x,t,\mu})$ and $\|\gamma(D)\|_t \geq \|\gamma(C)\|_t - \varepsilon$.*

Proof. For the pruposes of this argument, let us naturally identify $\mathbb{R}$ with the rotated line $\{re^{(t+\frac{1}{2})\pi i} : r \in \mathbb{R}\}$ which is the range of the map $\mathrm{proj}_t^\perp$.

Let $s_1, s_2 \in \mathbb{R}$ be such that $s_1 < s_2$ and $\mathrm{proj}_t^\perp(\gamma(C)) = [s_1, s_2]$ (and hence $\|\gamma(C)\|_t = s_2 - s_1$). We may assume that $\varepsilon < \frac{s_2 - s_1}{2}$. Let S' denote the narrower (closed) strip $(\mathrm{proj}_t^\perp)^{-1}([s_1 + \frac{\varepsilon}{2}, s_2 - \frac{\varepsilon}{2}]) \subset \mathrm{int}(S_j^{x,t,\mu})$. Then $C \cap \gamma^{-1}(S')$ must have a component D such that $\mathrm{proj}_t^\perp(\gamma(D)) = [s_1 + \frac{\varepsilon}{2}, s_2 - \frac{\varepsilon}{2}]$ (see e.g. Theorem 5.2 of [9]). This D is as desired. □

A real-valued function f is *lower semicontinuous* if $f^{-1}((\alpha, \infty))$ is open for every $\alpha \in \mathbb{R}$. Note that a lower semicontinuous function is Borel, hence (Lebesgue) integrable.

Lemma 3. *For a fixed generalized path $\gamma : X \to \mathbb{C}$, put $L(x, t, \mu) = L^{x,t,\mu}(\gamma)$. Then the function $L(x, t, \mu)$ from $[0, 1] \times [0, 1] \times (0, 1]$ to $\mathbb{R}$ is lower semicontinuous, hence integrable.*

Proof. Fix a number $\alpha \in \mathbb{R}$, and suppose $L^{x,t,\mu}(\gamma) > \alpha$. Choose N large enough so that $\sum_{n=0}^{N} \frac{\|\gamma(C_n^{x,t,\mu})\|_t}{2^n} > \alpha$.

For each $n \in \{0, 1, \ldots, N\}$ let $j(n)$ be such that $C_n^{x,t,\mu}$ is a component of $\gamma^{-1}(S_{j(n)}^{x,t,\mu})$. Then, by Lemma 2, for each n we can find a proper subcontinuum $D_n \subset C_n^{x,t,\mu}$ such that $\gamma(D_n)$ is contained in the interior of $S_{j(n)}^{x,t,\mu}$, and so that

$$\sum_{n=0}^{N} \frac{\|\gamma(D_n)\|_t}{2^n} > \alpha.$$

Let $\varepsilon_1 > 0$ be small enough so that if $|x' - x|, |t' - t|, |\mu' - \mu| < \varepsilon_1$, then $\gamma(D_n) \subset S_{j(n)}^{x',t',\mu'}$ for each $n \in \{0, 1, \ldots, N\}$, and moreover

$$\sum_{n=0}^{N} \frac{\|\gamma(D_n)\|_{t'}}{2^n} > \alpha. \tag{1}$$

For each pair of numbers $n_1 < n_2$ in $\{0, 1, \ldots, N\}$ with $j(n_1) = j(n_2)$, find an open set $A_{n_1,n_2} \subset X$ such that $C_{n_1}^{x,t,\mu} \subset A_{n_1,n_2} \subset \overline{A_{n_1,n_2}} \subset X \setminus C_{n_2}^{x,t,\mu}$ and $\partial A_{n_1,n_2} \cap \gamma^{-1}(S_{j(n_1)}^{x,t,\mu}) = \emptyset$; that is, $\gamma(\partial A_{n_1,n_2}) \cap S_{j(n_1)}^{x,t,\mu} = \emptyset$.

Let $0 < \varepsilon_2 < \varepsilon_1$ be small enough so that if $|x' - x|, |t' - t|, |\mu' - \mu| < \varepsilon_2$, then $\gamma(\partial A_{n_1,n_2}) \cap S_{j(n_1)}^{x',t',\mu'} = \emptyset$ for every pair of numbers $n_1 < n_2$ in $\{0, 1, \ldots, N\}$ with $j(n_1) = j(n_2)$. Since $\partial A_{n_1,n_2}$ separates D_{n_1} from D_{n_2} in X, it follows that D_{n_1} and D_{n_2} are contained in distinct components of $\gamma^{-1}(S_{j(n_1)}^{x',t',\mu'})$. Therefore, for such x', t', μ', by $(*)$ and (1) we have

$$L^{x',t',\mu'}(\gamma) \geq \sum_{n=0}^{N} \frac{\|\gamma(D_n)\|_{t'}}{2^n} > \alpha.$$

Thus, the set $\{(x,t,\mu) : L^{x,t,\mu}(\gamma) > \alpha\}$ is open in $[0,1]\times[0,1]\times(0,1]$, and so $L(x,t,\mu)$ is a lower semicontinuous function. $\square$

Thus the function len is well-defined. Observe that the set $\gamma(C_n^{x,t,\mu})$ is contained in some strip $S_j^{x,t,\mu}$ having width μ, hence $\|\gamma(C_n^{x,t,\mu})\|_t \leq \mu$. It follows that $L^{x,t,\mu}(\gamma) < 2\mu$, and therefore $\mathsf{len}(\gamma) < 1$.

It can easily be seen that $\mathsf{len}(\overline{0x}) \to 1$ as $x \to \infty$, $x \in \mathbb{R}$. It follows from Propositions 4(iii) and 6 below that if $\gamma_m : X_m \to \mathbb{C}$ is a sequence of generalized paths such that $\mathrm{diam}(\gamma_m(X_m)) \to \infty$ as $m \to \infty$, then $\mathsf{len}(\gamma_m) \to 1$ as $m \to \infty$.

On the other hand, if we define $\gamma_m : [0,1] \to \mathbb{C}$ by $\gamma_m(t) = e^{2\pi i m t}$, then $\mathsf{len}(\gamma_m) \to 1$ as $m \to \infty$, even though $\mathrm{diam}(\gamma_m([0,1])) = 2$ for all m.

3. Properties of the function len

The following basic properties follow immediately from the definition of the function len.

Proposition 4. *Let $\gamma : X \to \mathbb{C}$ be a generalized path.*

(i) $\mathsf{len}(\gamma) = 0$ *if and only if γ is a constant function.*
(ii) *If $h : Y \to X$ is a homeomorphism, then* $\mathsf{len}(\gamma \circ h) = \mathsf{len}(\gamma)$.
(iii) *If $\Phi : \mathbb{C} \to \mathbb{C}$ is an isometry, then* $\mathsf{len}(\Phi \circ \gamma) = \mathsf{len}(\gamma)$.

For the next properties, we need to consider a more restricted class of locally connected continua, namely dendrites. A *dendrite* is a locally connected continuum which contains no simple closed curve. A characteristic feature of dendrites is that they are hereditarily unicoherent; that is, given any two intersecting subcontinua A and B of a dendrite X, the intersection $A \cap B$ is connected.

Proposition 5. *Let X be a dendrite, and let $\gamma : X \to \mathbb{C}$ be a generalized path.*

(i) *If A is a subcontinuum of X, then* $\mathsf{len}(\gamma{\restriction_A}) \leq \mathsf{len}(\gamma)$. *Moreover,* $\mathsf{len}(\gamma{\restriction_A}) = \mathsf{len}(\gamma)$ *if and only if γ is constant on each component of $X \setminus A$.*
(ii) *If A, B are subcontinua of X with $A \cup B = X$, then*
$$\mathsf{len}(\gamma) \leq \mathsf{len}(\gamma{\restriction_A}) + \mathsf{len}(\gamma{\restriction_B}).$$

Proof. Fix x, t, μ, and for convenience denote $S_j^{x,t,\mu}$ and $C_n^{x,t,\mu}$ (defined as in Section 2) simply by S_j and C_n, respectively.

Let $A \subseteq X$ be a subcontinuum. Given $j \in \mathbb{Z}$ and a component C of $(\gamma{\restriction_A})^{-1}(S_j)$, there exists some n such that $C \subseteq C_n$. Since $C_n \cap A$ is connected (by hereditary unicoherence), it follows that $C = C_n \cap A$.

Therefore there exists an injective function σ from the non-negative integers to themselves such that $\langle C_{\sigma(n)} \cap A\rangle_{n=0}^{\infty}$ enumerates the collection of all components of the sets $(\gamma{\restriction_A})^{-1}(S_j^{x,t,\mu})$ $(j \in \mathbb{Z})$ which have non-degenerate image under the map $\mathrm{proj}_t^{\perp}$, so that $\|\gamma(C_{\sigma(n)} \cap A)\|_t \geq \|\gamma(C_{\sigma(n+1)} \cap A)\|_t$ for all n. Then

$$\begin{aligned} L^{x,t,\mu}(\gamma{\restriction_A}) &= \sum_{n=0}^{\infty} \frac{\|\gamma(C_{\sigma(n)} \cap A)\|_t}{2^n} \\ &\leq \sum_{n=0}^{\infty} \frac{\|\gamma(C_{\sigma(n)})\|_t}{2^n} \\ &\leq L^{x,t,\mu}(\gamma) \qquad \text{(by the observation } (*)). \end{aligned}$$

Since this holds for all x, t, μ, we have established the first statement of (i).

For the second statement of (i), suppose γ is non-constant on some component K of $X \smallsetminus A$. The intersection $\overline{K} \cap A$ consists of a single point (see e.g. 10.9 and 10.24 of [9]). Let $\{p\} = \overline{K} \cap A$, and let $q \in K$ be such that $\gamma(p) \neq \gamma(q)$. There is a positive measure set of parameters x, t, μ and an integer $j \in \mathbb{Z}$ for which $\gamma(q) \in \mathrm{int}(S_j^{x,t,\mu})$ and $\gamma(p) \notin S_j^{x,t,\mu}$. For such x, t, μ, j, there is a component of $\gamma^{-1}(S_j^{x,t,\mu})$ contained in K, which contributes positively to the sum $L^{x,t,\mu}(\gamma)$, thereby making it larger than $L^{x,t,\mu}(\gamma{\restriction_A})$. It follows that $\mathsf{len}(\gamma) > \mathsf{len}(\gamma{\restriction_A})$. The converse implication is immediate.

Now suppose $A, B \subseteq X$ are subcontinua with $A \cup B = X$. As above, for any $j \in \mathbb{Z}$, each component of $(\gamma{\restriction_A})^{-1}(S_j)$ (respectively $(\gamma{\restriction_B})^{-1}(S_j)$) has the form $C_n \cap A$ (respectively $C_n \cap B$) for some n.

Let $\langle n(\alpha)\rangle_{\alpha=0}^{\infty}$ and $\langle m(\beta)\rangle_{\beta=0}^{\infty}$ be the strictly increasing sequences of non-negative integers such that $\langle C_{n(\alpha)} \cap A\rangle_{\alpha=0}^{\infty}$ enumerates the collection of all components of the sets $(\gamma{\restriction_A})^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\mathrm{proj}_t^{\perp} \circ \gamma$, and $\langle C_{m(\beta)} \cap B\rangle_{\beta=0}^{\infty}$ enumerates the collection of all components of the sets $(\gamma{\restriction_B})^{-1}(S_j^{x,t,\mu})$ ($j \in \mathbb{Z}$) which have non-degenerate image under the map $\mathrm{proj}_t^{\perp} \circ \gamma$. Note that these enumerations are not necessarily ordered according to the sizes of the images under $\mathrm{proj}_t^{\perp} \circ \gamma$.

For any n, we clearly have $\|\gamma(C_n)\|_t \leq \|\gamma(C_n \cap A)\|_t + \|\gamma(C_n \cap B)\|_t$. Therefore

$$\begin{aligned}
L^{x,t,\mu}(\gamma) &= \sum_{n=0}^{\infty} \frac{\|\gamma(C_n)\|_t}{2^n} \\
&\leq \sum_{n=0}^{\infty} \frac{\|\gamma(C_n \cap A)\|_t}{2^n} + \sum_{n=0}^{\infty} \frac{\|\gamma(C_n \cap B)\|_t}{2^n} \\
&= \sum_{\alpha=0}^{\infty} \frac{\|\gamma(C_{n(\alpha)} \cap A)\|_t}{2^{n(\alpha)}} + \sum_{\beta=0}^{\infty} \frac{\|\gamma(C_{m(\beta)} \cap B)\|_t}{2^{m(\beta)}} \\
&\leq \sum_{\alpha=0}^{\infty} \frac{\|\gamma(C_{n(\alpha)} \cap A)\|_t}{2^{\alpha}} + \sum_{\beta=0}^{\infty} \frac{\|\gamma(C_{m(\beta)} \cap B)\|_t}{2^{\beta}} \qquad (\text{since } \alpha \leq n(\alpha),\ \beta \leq m(\beta)) \\
&\leq L^{x,t,\mu}(\gamma{\restriction_A}) + L^{x,t,\mu}(\gamma{\restriction_B}) \qquad (\text{by the observation } (*)).
\end{aligned}$$

Since this holds for all x, t, μ, we have established (ii). □

To see that the assumption that X is a dendrite in Proposition 5 is necessary, consider the identity function $\mathrm{id}_{\mathbb{D}}$ on the unit disk $\mathbb{D} \subset \mathbb{C}$ with boundary circle $\mathbb{S}^1$. It is not difficult to see that $\mathsf{len}(\mathrm{id}_{\mathbb{D}}) < \mathsf{len}(\mathrm{id}_{\mathbb{S}^1})$.

Moreover, consider the embedding O of the circle $\mathbb{S}^1$ depicted in Figure 1. Let $\gamma : [0, 1] \to O$ be a path which goes exactly once around the circle O, starting and ending at the indicated point p, and otherwise one-to-one. We claim that $\mathsf{len}(\gamma) < \mathsf{len}(\mathrm{id}_O)$, which can be argued as follows:

Given a strip $S_j^{x,t,\mu}$ containing the point p, the component C of $O \cap S_j^{x,t,\mu}$ containing p corresponds to two components $[0, c]$ and $[d, 1]$ of $\gamma^{-1}(S_j^{x,t,\mu})$. For nearly horizontal strips (i.e. for t values close to 0 or 1) the sets $\mathrm{proj}_t^{\perp}(\gamma([0, c]))$ and $\mathrm{proj}_t^{\perp}(\gamma([d, 1]))$ may overlap; however, because of the oscillation up and down on the left and right sides of the circle O, for such parameters x, t, μ there are

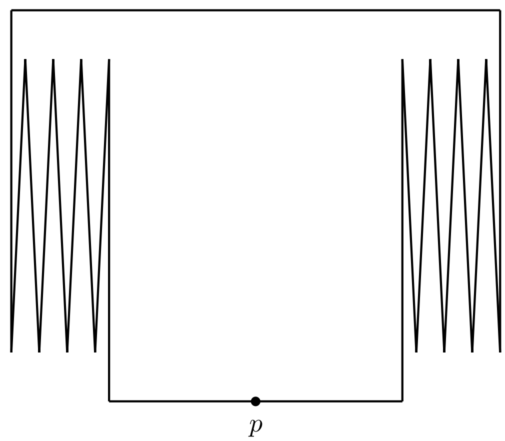

Figure 1. A particular embedding of the circle in the plane.

many other components of $[0,1] \cap \gamma^{-1}(S_j^{x,t,\mu})$ and of $O \cap S_j^{x,t,\mu}$ $(j \in \mathbb{Z})$ with large projections, hence the weighted sums $L^{x,t,\mu}(\mathrm{id}_O)$ and $L^{x,t\mu}(\gamma)$ will differ only very slightly. For all other values of x, t, μ, the sets $\mathrm{proj}_t^{\perp}(\gamma([0,c]))$ and $\mathrm{proj}_t^{\perp}(\gamma([d,1]))$ share only the point $\mathrm{proj}_t^{\perp}(p)$, and one of them will be added with a smaller weight in the sum $L^{x,t,\mu}(\gamma)$ than that of C in $L^{x,t,\mu}(\mathrm{id}_O)$. In particular, this is so for values of x, t, μ for which the strips $S_j^{x,t,\mu}$ are wide and nearly vertical, and for these values resulting difference between $L^{x,t,\mu}(\mathrm{id}_O)$ and $L^{x,t,\mu}(\gamma)$ will be more pronounced due to the small number of terms in these sums. Thus, with an appropriate amount of oscillation, we obtain that $\mathsf{len}(\gamma) < \mathsf{len}(\mathrm{id}_O)$.

Now if we let A be a very small arc in O containing the point p and such that $\mathsf{len}(\mathrm{id}_A) < \mathsf{len}(\mathrm{id}_O) - \mathsf{len}(\gamma)$, and let $A' = \overline{O \setminus A}$, then it follows that $\mathsf{len}(\mathrm{id}_O) > \mathsf{len}(\mathrm{id}_A) + \mathsf{len}(\mathrm{id}_{A'})$.

Proposition 6. *Let $z_1, z_2 \in \mathbb{C}$. If $\gamma : X \to \mathbb{C}$ is any generalized path such that $z_1, z_2 \in \gamma(X)$, then $\mathsf{len}(\overline{z_1 z_2}) \leq \mathsf{len}(\gamma)$. Moreover, if $\gamma(X)$ is not the straight line segment joining z_1 and z_2, or if $\gamma^{-1}(w)$ is disconnected for some w on the straight line segment between z_1 and z_2, then $\mathsf{len}(\overline{z_1 z_2}) < \mathsf{len}(\gamma)$.*

Proposition 6 can be proved directly from the definition of the function len, and we leave this to the reader.

Proposition 7. *Suppose $\langle \gamma_m \rangle_{m=0}^{\infty}$ is a sequence of generalized paths $X \to \mathbb{C}$ converging uniformly to $\gamma : X \to \mathbb{C}$. Then $\mathsf{len}(\gamma_m) \to \mathsf{len}(\gamma)$ as $m \to \infty$.*

Proof. Suppose $\alpha < \mathsf{len}(\gamma) < \beta$.

A simple modification of the proof of Lemma 3 shows that for large m, $\mathsf{len}(\gamma_m) > \alpha$. Thus it remains to show $\mathsf{len}(\gamma_m) < \beta$ for large m.

Fix a countable dense set $\{q_k\}_{k=0}^{\infty} \subset X$. Given $k \neq l$ and $j \in \mathbb{Z}$, let

$$
\begin{aligned}
B_{kl}^j = \{(x,t,\mu) \in [0,1] \times [0,1] \times (0,1] :& \\
&\text{there is a continuum } C \subseteq \gamma^{-1}(S_j^{x,t,\mu}) \text{ with } q_k, q_l \in C \text{ and} \\
&\text{for every such } C \text{ we have } \gamma(C) \cap \partial S_j^{x,t,\mu} \neq \emptyset\}
\end{aligned}
$$

and let $B = \bigcup_{\substack{k\neq l\\ j\in\mathbb{Z}}} B^j_{kl}$. It is easy to see that $([0,1]\times[0,1]\times[0,1]) \smallsetminus B^j_{kl}$ is open, and so B is F_σ, hence measurable.

Claim 7.1. B has measure zero.

Proof of Claim 7.1. Fix $k \neq l$ and $j \in \mathbb{Z}$. It will be convenient to change variables from (x,t,μ) to (z,t,μ) so that for any fixed rotation angle t and translation parameter z, as the strip width μ shrinks, the j-th strip itself shrinks inwards, nesting down on a line.

Given $(x,t,\mu) \in [0,1]\times[0,1]\times(0,1]$, let $z = \mu(x+\frac{1}{2}+j) \in (-\infty,\infty)$, and define $\Phi(x,t,\mu) = (z,t,\mu)$.

Observe that for (z,t,μ) in the image of Φ, $\Phi^{-1}(z,t,\mu) = (\frac{z}{\mu} - \frac{1}{2} - j, t, \mu)$. Thus $\Phi(B^j_{kl})$ is contained in the set

$$\begin{aligned} B' = \{(z,t,\mu) : \text{ there is a continuum } C \subseteq \gamma^{-1}(T^{z,t,\mu}) \text{ with } q_k, q_l \in C \text{ and} \\ \text{for every such } C \text{ we have } \gamma(C) \cap \partial T^{z,t,\mu} \neq \emptyset\} \end{aligned}$$

where $T^{z,t,\mu} = \mu e^{t\pi i}(S_j + i(\frac{z}{\mu} - \frac{1}{2} - j)) = e^{t\pi i}(\mu(S_j - \frac{1}{2}i - ji) + iz)$. Observe that the strip $T^{z,t,\mu}$ is centered about the line $e^{t\pi i}(\mathbb{R}+iz)$, and if $\mu' < \mu$, then $T^{z,t,\mu'}$ is contained in the interior of $T^{z,t,\mu}$. Thus for any fixed z,t, there can be at most one μ for which $(z,t,\mu) \in B'$. By Fubini's theorem, this implies B' has measure zero. Since $\Phi(B^j_{kl}) \subseteq B'$, we have that $\Phi(B^j_{kl})$ has measure zero as well.

A straightforward calculation shows that Φ is a C^1-diffeomorphism on $[0,1]\times[0,1]\times(0,1]$ with Jacobian equal to μ. Thus by the change of variables theorem [6, Theorem 2.47], the measure of $\Phi(B^j_{kl})$ is equal to

$$\iiint_{B^j_{kl}} \mu \, dx\, dt\, d\mu.$$

Since $\mu > 0$ and $\Phi(B^j_{kl})$ has measure zero, it follows that B^j_{kl} has measure zero as well. Since $B = \bigcup_{\substack{k\neq l\\ j\in\mathbb{Z}}} B^j_{kl}$, the claim follows. $\square$(Claim 7.1)

Claim 7.2. Given $(x_0,t_0,\mu_0) \in ([0,1]\times[0,1]\times(0,1]) \smallsetminus B$ and $\varepsilon > 0$, there exists $\delta > 0$ and an integer M such that if $|x - x_0| < \delta$, $|t - t_0| < \delta$, $|\mu - \mu_0| < \delta$, and $m \geq M$, then $L^{x,t,\mu}(\gamma_m) < L^{x_0,t_0,\mu_0}(\gamma) + \varepsilon$.

Proof of Claim 7.2. For $j \in \mathbb{Z}$, let S'_j denote the narrower (closed) strip obtained from S^{x_0,t_0,μ_0}_j by moving the boundary lines in towards the middle a distance of $\frac{\varepsilon}{20}$ each.

Let $\langle C_n\rangle_{n=0}^\infty$ enumerate the collection of all components of the sets $\gamma^{-1}(S^{x_0,t_0,\mu_0}_j)$ $(j \in \mathbb{Z})$ which have non-degenerate image under the map $\mathrm{proj}^\perp_t$, ordered so that $\|\gamma(C_n)\|_{t_0} \geq \|\gamma(C_{n+1})\|_{t_0}$ for all n. For each n, let $j(n)$ be the integer such that $\gamma(C_n) \subset S^{x_0,t_0,\mu_0}_{j(n)}$. By Lemma 1, there are only finitely many components $C_0,\ldots,C_N$ such that $\gamma(C_n)$ meets the narrower strip $S'_{j(n)}$, for $0 \leq n \leq N$.

Fix some n with $0 \leq n \leq N$. Let $U_0,\ldots,U_r$ be a finite cover of $C_n \cap \gamma^{-1}(S'_{j(n)})$ by connected open subsets of X whose closures are mapped by γ into the interior of $S^{x_0,t_0,\mu_0}_{j(n)}$. Let k be such that $q_k \in U_0$, and for each $1 \leq i \leq r$ let $l(i)$ be such that $q_{l(i)} \in U_i$. Then for each $1 \leq i \leq r$, since $(x_0,t_0,\mu_0) \notin B^{j(n)}_{k\,l(i)}$ and C_n is a

continuum in $\gamma^{-1}(S^{x_0,t_0,\mu_0}_{j(n)})$ containing q_k and $q_{l(i)}$, there exists a continuum K_i containing q_k and $q_{l(i)}$ which is mapped by γ into the interior of the strip $S^{x_0,t_0,\mu_0}_{j(n)}$. Let $C'_n = \overline{U_0} \cup \bigcup_{1\le i\le r}(\overline{U_i} \cup K_i)$. Then C'_n is a continuum which is mapped by γ into the interior of the strip $S^{x_0,t_0,\mu_0}_{j(n)}$ and such that $C_n \cap \gamma^{-1}(S'_j) \subseteq C'_n \subset C_n$.

Having done this for each $0 \le n \le N$, let $\delta > 0$ be small enough and let M be large enough so that if $|x - x_0| < \delta$, $|t - t_0| < \delta$, $|\mu - \mu_0| < \delta$, and $m \ge M$, then for each $0 \le n \le N$ we have:

(i) $\gamma_m(C'_n)$ is contained in the interior of the strip $S^{x,t,\mu}_{j(n)}$,
(ii) $\|\gamma_m(C'_n)\|_t < \|\gamma(C'_n)\|_{t_0} + \frac{\varepsilon}{4}$, and
(iii) if $A \subset X$ with $\gamma(A)$ contained in between two consecutive narrowed strips S'_j and S'_{j+1}, then $\|\gamma_m(A)\|_t < \frac{\varepsilon}{8}$.

Note that if $0 \le n \le N$ and if C is the component of $\gamma_m^{-1}(S^{x,t,\mu}_{j(n)})$ containing C'_n, then C consists of C'_n plus some part which γ maps in between $S'_{j(n)}$ and $S'_{j(n)-1}$, and some part which γ maps in between $S'_{j(n)}$ and $S'_{j(n)+1}$. Therefore, by (ii) and (iii) we have

$$\|\gamma_m(C)\|_t < \|\gamma(C'_n)\|_{t_0} + \tfrac{\varepsilon}{4} + 2\cdot\tfrac{\varepsilon}{8} = \|\gamma(C'_n)\|_{t_0} + \tfrac{\varepsilon}{2}.$$

Every other component $\tilde{C}$ of $\gamma_m^{-1}(S^{x,t,\mu}_j)$ satisfies $\|\gamma_m(\tilde{C})\|_t < \frac{\varepsilon}{8}$ by (iii). It follows that

$$\begin{aligned} L^{x,t,\mu}(\gamma_m) &< \sum_{n=0}^{N} \frac{\|\gamma(C'_n)\|_{t_0} + \frac{\varepsilon}{2}}{2^n} + \sum_{n=N+1}^{\infty} \frac{\frac{\varepsilon}{8}}{2^n} \\ &< \sum_{n=0}^{N} \frac{\|\gamma(C'_n)\|_{t_0}}{2^n} + \varepsilon \\ &\le L^{x_0,t_0,\mu_0}(\gamma) + \varepsilon. \end{aligned}$$

□(Claim 7.2)

We are now ready to show that $\mathsf{len}(\gamma_m) < \beta$ for large m.

Recalling that $L^{x,t,\mu}(\gamma) < 2\mu \le 2$, choose a step function $\psi = 2 - \sum_{i=0}^{k} c_i\,\chi_{A_i}$, where the $A_i \subset [0,1]\times[0,1]\times(0,1]$ are pairwise disjoint compact sets and χ_{A_i} is the characteristic function of the set A_i, with

$$L^{x,t,\mu}(\gamma) \le \psi(x,t,\mu) \quad \text{for all } x,t,\mu$$

and

$$\int_0^1\int_0^1\int_0^1 \psi(x,t,\mu)\,dx\,dt\,d\mu < \beta.$$

Let $\eta = \beta - \int_0^1\int_0^1\int_0^1 \psi\,dx\,dt\,d\mu > 0$. By Claim 7.1, we can find a compact set $\Omega \subset ([0,1]\times[0,1]\times(0,1]) \setminus B$ of measure $\ge 1 - \frac{\eta}{4}$.

Using Claim 7.2 and compactness of the sets $A_i \cap \Omega$, we can find M_i large enough so that if $m \ge M_i$, then $L^{x,t,\mu}(\gamma_m) < \psi(x,t,\mu) + \frac{\eta}{4}$ for all $(x,t,\mu) \in A_i \cap \Omega$. Letting $M = \max_i M_i$, it follows that if $m \ge M$, then

$$\begin{aligned}
\mathsf{len}(\gamma_m) &= \int_0^1 \int_0^1 \int_0^1 L^{x,t,\mu}(\gamma_m)\,dx\,dt\,d\mu \\
&\leq \iiint\limits_{\Omega} L^{x,t,\mu}(\gamma_m)\,dx\,dt\,d\mu + 2\cdot\tfrac{\eta}{4} \\
&\leq \iiint\limits_{\Omega} \left(\psi(x,t,\mu) + \tfrac{\eta}{4}\right)\,dx\,dt\,d\mu + 2\cdot\tfrac{\eta}{4} \\
&\leq (\beta-\eta) + \tfrac{\eta}{4} + 2\cdot\tfrac{\eta}{4} \\
&< \beta.
\end{aligned}$$

□

Corollary 8. *Given a homotopy $h : X \times [0,1] \to \mathbb{C}$, the function $t \mapsto \mathsf{len}(h_t)$ is continuous, where $h_t : X \to \mathbb{C}$ is defined by $h_t(x) = h(x,t)$.*

A consequence of Proposition 7 is that for a path $\gamma : [a,b] \to \mathbb{C}$, $\mathsf{len}(\gamma)$ is small if and only if $\mathrm{diam}(\gamma([a,b]))$ is small. The following elementary estimate is more precise, and follows from the definition of the function len:

$(**)$: If $\mathrm{diam}(\gamma([a,b])) \leq \frac{1}{2}$, then

$$\tfrac{1}{2\pi}\,\mathrm{diam}(\gamma([a,b])) \leq \mathsf{len}(\gamma) \leq 2\,\mathrm{diam}(\gamma([a,b])).$$

4. Parameterization by len

One can deduce from Propositions 5 (i) and 7 that given a path $\gamma : [a,b] \to \mathbb{C}$, the function $[a,b] \to [0,1)$ defined by $t \mapsto \mathsf{len}(\gamma\restriction_{[a,t]})$ is continuous and non-decreasing. It follows that there is a *standard representation* $\tilde{\gamma} : [0, \mathsf{len}(\gamma)] \to \mathbb{C}$ of γ, also called the parameterization of γ by path length, defined as follows: given $s \in [0, \mathsf{len}(\gamma)]$, $\tilde{\gamma}(s) = \gamma(t)$, where $t \in [a,b]$ is such that $\mathsf{len}(\gamma\restriction_{[a,t]}) = s$. Note that this value t may not be unique, but by Proposition 5 (i), the point $\gamma(t)$ is uniquely determined by s. One can easily check that $\tilde{\gamma}$ is a path (i.e. is a continuous function), and $\mathsf{len}(\tilde{\gamma}\restriction_{[0,s]}) = s$ for any $s \in [0, \mathsf{len}(\gamma)]$. However, note that in general $\mathsf{len}(\tilde{\gamma}\restriction_{[s_1,s_2]}) \neq s_2 - s_1$ when $0 < s_1 < s_2 \leq 1$.

For the Euclidean path length, such a representation is only available for rectifiable paths, i.e. those paths with finite Euclidean length.

A classical result from metric geometry (see e.g. [3]) is that if $L > 0$ and $\langle \gamma_m \rangle_{m=0}^{\infty}$ is a sequence of paths in a bounded set, with Euclidean path lengths $\leq L$, and if $\tilde{\gamma}_m : [0,1] \to \mathbb{C}$ is the parameterization of γ_m by Euclidean path length (with domain linearly rescaled to $[0,1]$), then the sequence $\langle \tilde{\gamma}_m \rangle_{m=0}^{\infty}$ has a subsequence which converges uniformly to a path of finite Euclidean length. This reparameterization is necessary, as standard examples show (consider e.g. $\gamma_m : [0,1] \to [0,1]$ defined by $\gamma_m(s) = s^m$).

We will now prove a version of this result for the function len, where the uniform bound on length assumption is replaced by a weaker restriction on the number of long sections of the paths. Moreover, we prove that this condition is in fact a characterization of those families of paths which can be reparameterized so as to be equicontinuous.

Theorem 9. *Let $\mathcal{F}$ be a family of paths $[0,1] \to \mathbb{C}$. Suppose that*

(†): *for each $\varepsilon > 0$, there is a positive integer N such that for every $\gamma \in \mathcal{F}$, there is no collection of more than N pairwise disjoint subintervals of $[0,1]$ whose images under γ have diameters $\geq \varepsilon$.*

Given $\gamma \in \mathcal{F}$, let $\tilde{\gamma} : [0,1] \to \mathbb{C}$ be the standard representation of γ, with domain linearly rescaled from $[0, \mathsf{len}(\gamma)]$ to $[0,1]$. Then the family $\tilde{\mathcal{F}} = \{\tilde{\gamma} : \gamma \in \mathcal{F}\}$ is equicontinuous. In particular, if there a compact set $K \subset \mathbb{C}$ such that $\gamma([0,1]) \subseteq K$ for every $\gamma \in \mathcal{F}$, then every sequence $\langle \tilde{\gamma}_m \rangle_{m=0}^{\infty} \subseteq \tilde{\mathcal{F}}$ has a subsequence $\langle \tilde{\gamma}_{m_k} \rangle_{k=0}^{\infty}$ which converges uniformly to a path $\tilde{\gamma}_\infty : [0,1] \to K$, and this function $\tilde{\gamma}_\infty$ is the standard representation of the limit path.

Conversely, if $\mathcal{F}$ is an equicontinuous family of paths, then it satisfies the property (†).

Proof. Fix $\varepsilon > 0$. Let N be such that for every $\gamma \in \mathcal{F}$, there is no collection of more than N pairwise disjoint subintervals of $[0,1]$ whose images under γ have diameters $\geq \frac{\varepsilon}{16}$. Let $\delta = \frac{\varepsilon^2}{2^{N+8}}$.

Suppose for a contradiction that for some $\gamma \in \mathcal{F}$ there exist $0 \leq s_1 < s_2 \leq 1$ with $s_2 - s_1 < \delta$ and $d(\tilde{\gamma}(s_1), \tilde{\gamma}(s_2)) \geq \varepsilon$. Note that since $\tilde{\gamma}$ is the standard representation of γ with domain scaled by a factor $\frac{1}{\mathsf{len}(\gamma)}$, and $\mathsf{len}(\gamma) < 1$, we have

$$\begin{aligned}\mathsf{len}(\tilde{\gamma}\restriction_{[0,s_2]}) &= s_2 \cdot \mathsf{len}(\gamma) \\ &< s_1 \cdot \mathsf{len}(\gamma) + \delta \cdot \mathsf{len}(\gamma) \\ &< \mathsf{len}(\tilde{\gamma}\restriction_{[0,s_1]}) + \delta.\end{aligned}$$

Let $t_0 \in [0,1]$ be such that the line $\{re^{t_0 \pi i} : r \in \mathbb{R}\}$ is orthogonal to the segment $\overline{\tilde{\gamma}(s_1)\tilde{\gamma}(s_2)}$. Define $W \subset [0,1] \times [0,1] \times (0,1]$ by

$$W = [0,1] \times [t_0 - \tfrac{1}{4}, t_0 + \tfrac{1}{4}] \times [\tfrac{\varepsilon}{8}, \tfrac{\varepsilon}{4}],$$

where the interval $[t_0 - \frac{1}{4}, t_0 + \frac{1}{4}]$ should be considered reduced mod 1 (i.e. it represents the set of all $t \in [0,1]$ such that one of $|t - t_0|$, $|t - (t_0 - 1)|$, or $|t - (t_0 + 1)|$ is $\leq \frac{1}{4}$). Note that for any $(x,t,\mu) \in W$, any strip $S_j^{x,t,\mu}$ ($j \in \mathbb{Z}$) covers less than half of the line segment $\overline{\tilde{\gamma}(s_1)\tilde{\gamma}(s_2)}$. It follows that for all such x, t, μ, we have that for some $j \in \mathbb{Z}$, there is a component C of $\tilde{\gamma}_m^{-1}(S_j^{x,t,\mu})$ such that $C \subset (s_1, s_2)$ and $\|\tilde{\gamma}_m(C)\|_t = \mu \geq \frac{\varepsilon}{8}$. This component C contributes an extra term to the sum $L^{x,t,\mu}(\tilde{\gamma}\restriction_{[0,s_2]})$ which is not present in $L^{x,t,\mu}(\tilde{\gamma}\restriction_{[0,s_1]})$. Since there are at most $N-1$ terms $\frac{\|\tilde{\gamma}(C')\|_t}{2^k}$ in the sum $L^{x,t,\mu}(\tilde{\gamma}\restriction_{[0,s_1]})$ with $\|\tilde{\gamma}(C')\|_t = \mu$, and the next term has size $\leq \frac{\varepsilon/16}{2^N}$, we have that

$$\begin{aligned}L^{x,t,\mu}(\tilde{\gamma}\restriction_{[0,s_2]}) &\geq L^{x,t,\mu}(\tilde{\gamma}\restriction_{[0,s_1]}) + \frac{\varepsilon/16}{2^N} \\ &= L^{x,t,\mu}(\tilde{\gamma}\restriction_{[0,s_1]}) + \frac{\varepsilon}{2^{N+4}}.\end{aligned}$$

Noting that the measure of W is $1 \cdot \frac{1}{2} \cdot (\frac{\varepsilon}{4} - \frac{\varepsilon}{8}) = \frac{\varepsilon}{16}$, it follows that

$$\begin{aligned}\mathsf{len}(\tilde{\gamma}\restriction_{[0,s_2]}) &\geq \mathsf{len}(\tilde{\gamma}\restriction_{[0,s_1]}) + \frac{\varepsilon}{2^{N+4}} \cdot \frac{\varepsilon}{16} \\ &= \mathsf{len}(\tilde{\gamma}\restriction_{[0,s_1]}) + \delta.\end{aligned}$$

But this contradicts the assumption that $s_2 - s_1 < \delta$.

Thus for every $\gamma \in \mathcal{F}$, if $0 \leq s_1 < s_2 \leq 1$ with $s_2 - s_1 < \delta$, then $d(\tilde{\gamma}(s_1), \tilde{\gamma}(s_2)) < \varepsilon$.

If $K \subset \mathbb{C}$ is compact and $\gamma([0,1]) \subseteq K$ for all $\gamma \in \mathcal{F}$, then by the Arzelà-Ascoli theorem, given any sequence $\langle\tilde{\gamma}_m\rangle_{m=0}^{\infty} \subseteq \tilde{\mathcal{F}}$, there is a subsequence $\langle\tilde{\gamma}_{m_k}\rangle_{k=0}^{\infty}$ which converges uniformly to a path $\tilde{\gamma}_\infty : [0,1] \to K$. That $\tilde{\gamma}_\infty$ is the standard representation of this limit path follows from Proposition 7 and the fact that $\tilde{\gamma}_m$ is the standard representation of γ_m for each m.

For the converse, suppose there is some $\varepsilon > 0$ such that for any positive integer N, there exists a path $\gamma_N \in \mathcal{F}$ and a collection of N disjoint subintervals of $[0,1]$ whose images under γ_N have diameters $\geq \varepsilon$. Note that at least one of these subintervals must have width $\leq \frac{1}{N}$; denote it by J_N.

Let $s \in [0,1]$ be an accumulation point of the centers of the intervals J_N, $N = 0,1,2,\ldots$. Then for any $\delta > 0$, there is some N such that $J_N \subset (s-\delta, s+\delta)$, and hence $\gamma_N((s-\delta, s+\delta))$ has diameter $\geq \varepsilon$. Thus $\mathcal{F}$ is not equicontinuous. $\square$

Recall from the end of Section 3 that if $\gamma : [a,b] \to \mathbb{C}$ is a path, then

$$\tfrac{1}{2\pi}\operatorname{diam}(\gamma([a,b])) \leq \mathsf{len}(\gamma) \leq 2\operatorname{diam}(\gamma([a,b]))$$

whenever $\operatorname{diam}(\gamma([a,b])) \leq \frac{1}{2}$. Therefore, the hypothesis (†) in Theorem 9 can be replaced by the analogous statement involving len instead of diam, which yields the following result.

Corollary 10. *Let $\mathcal{F}$ be a family of paths $[0,1] \to \mathbb{C}$. Suppose that*

> (†′): *for each $\varepsilon > 0$, there is a positive integer N such that for every $\gamma \in \mathcal{F}$, there is no collection of more than N pairwise disjoint subintervals J of $[0,1]$ for which $\mathsf{len}(\gamma{\restriction}J) \geq \varepsilon$.*

Given $\gamma \in \mathcal{F}$, let $\tilde{\gamma} : [0,1] \to \mathbb{C}$ be the standard representation of γ, with domain linearly rescaled from $[0, \mathsf{len}(\gamma)]$ to $[0,1]$. Then the family $\tilde{\mathcal{F}} = \{\tilde{\gamma} : \gamma \in \mathcal{F}\}$ is equicontinuous. In particular, if there a compact set $K \subset \mathbb{C}$ such that $\gamma([0,1]) \subseteq K$ for every $\gamma \in \mathcal{F}$, then every sequence $\langle\tilde{\gamma}_m\rangle_{m=0}^{\infty} \subseteq \tilde{\mathcal{F}}$ has a subsequence $\langle\tilde{\gamma}_{m_k}\rangle_{k=0}^{\infty}$ which converges uniformly to a path $\tilde{\gamma}_\infty : [0,1] \to K$, and this function $\tilde{\gamma}_\infty$ is the standard representation of the limit path.

Conversely, if $\mathcal{F}$ is an equicontinuous family of paths, then it satisfies the property (†′).

5. Shortest paths

In [2], Bourgin and Renz proved that given a simply connected plane domain U with simple closed curve boundary, between any two points in $\overline{U}$ there is a unique shortest path in $\overline{U}$. In Theorem 11, we extend this result to paths in multiply connected domains, and provide an alternative characterization of a shortest path.

Fix a connected open set $\Omega \subset \mathbb{C}$, and let $p, q \in \overline{\Omega}$. By a *path in Ω joining p and q*, we mean a path $\gamma : [0,1] \to \mathbb{C}$ such that $\gamma(0) = p$, $\gamma(1) = q$, and $\gamma(s) \in \Omega$ for all $s \in (0,1)$.

We will consider homotopies $h : [0,1] \times [0,1] \to \mathbb{C}$ such that for each $t \in [0,1]$, the path $h_t : [0,1] \to \mathbb{C}$ defined by $h_t(s) = h(s,t)$ is a path in Ω joining p and q. Given a path γ in Ω joining p and q, let $[\gamma]$ denote the set of all paths homotopic to γ under such homotopies. Denote by $\overline{[\gamma]}$ the set of all paths in the closure of $[\gamma]$ (in the uniform metric). Note that paths in $\overline{[\gamma]}$ may meet the boundary $\partial\Omega$ in several places.

Our next goal is to prove the existence of a unique *shortest* path in the closure of any given homotopy class. We wish to include the possibility that Ω, p, q are such that there are no paths of finite Euclidean length in Ω joining p and q. To this end, we introduce a more general definition of a shortest path.

Definition. Let γ be a path in Ω. A path $\gamma_0 \in \overline{[\gamma]}$ is called a *shortest path in* $\overline{[\gamma]}$ if the following property holds:

> Given $s_1, s_2 \in [0,1]$ with $s_1 < s_2$, let γ_0' be the path defined by $\gamma_0'(s) = \gamma_0(s)$ for $s \notin [s_1, s_2]$, and $\gamma_0' \restriction_{[s_1,s_2]}$ parameterizes the straight line segment $\overline{\gamma_0(s_1)\gamma_0(s_2)}$. If $\gamma_0' \in \overline{[\gamma]}$, then $\gamma_0 = \gamma_0'$ (up to reparameterization).

Observe that if $\gamma_0 \in \overline{[\gamma]}$ and $\mathsf{len}(\gamma_0)$ is smallest among all paths in $\overline{[\gamma]}$, then γ_0 is a shortest path in $\overline{[\gamma]}$ (by Proposition 6). Conversely, by the uniqueness part of Theorem 11 below, it follows that the shortest path in $\overline{[\gamma]}$ also has smallest len among all paths in $\overline{[\gamma]}$ (otherwise, we could perform the construction in the proof of Theorem 11 beginning with a path in $[\gamma]$ with smaller len, thus obtaining a shortest path with smaller len, which would contradict the uniqueness).

Similar considerations show that the shortest path γ_0 in $\overline{[\gamma]}$ is characterized as the path in $\overline{[\gamma]}$ of smallest Euclidean length; or, if there is no path in $\overline{[\gamma]}$ of finite Euclidean length, then it has the property that for any $0 < t_1 < t_2 < 1$, the path $\gamma_0 \restriction_{[t_1,t_2]}$ has smallest (finite) Euclidean length among all paths homotopic to $\gamma_0 \restriction_{[t_1,t_2]}$ (relative to the endpoints $\gamma_0(t_1)$ and $\gamma_0(t_2)$).

Theorem 11. *Let $\Omega \subset \mathbb{C}$ be a connected open set, $p, q \in \overline{\Omega}$, and let γ be a path in Ω joining p and q. There exists a unique (up to parameterization) shortest path $\gamma_0 \in \overline{[\gamma]}$.*

Proof. Let Δ be a large compact disk in $\mathbb{C}$ containing the entire path $\gamma([0,1])$.

We may assume that $\partial\Omega$ contains more than one point in $\mathbb{C}$, as the claim is trivial otherwise.

Then the universal covering space of Ω is the unit disk $\mathbb{D}$, with analytic covering map $\Phi : \mathbb{D} \to \Omega$ (see e.g. [5]). For the basic theory of covering spaces, we refer the reader to [7]. It is well known that Φ may be extended to the boundary $\partial\mathbb{D}$ at points corresponding to accessible points of $\partial\Omega$, though this extension is not in general continuous (in an unrestricted sense) up to the boundary. Below, Φ will refer to this extension.

Given a path $\lambda : [0,1] \to \overline{\Omega}$, by a *lift* of λ, we mean a path $\hat{\lambda} : [0,1] \to \overline{\mathbb{D}}$ such that $\Phi \circ \hat{\lambda} = \lambda$.

The path γ lifts to a path $\hat{\gamma} : [0,1] \to \overline{\mathbb{D}}$ such that $\hat{\gamma}(0) = \hat{p} \in \overline{\mathbb{D}}$, $\hat{\gamma}(1) = \hat{q} \in \overline{\mathbb{D}}$, and $\hat{\gamma}((0,1)) \subset \mathbb{D}$. If $\hat{p} = \hat{q}$, then γ is homotopic to a constant path, and the result follows immediately. Hence we may assume $\hat{p} \neq \hat{q}$.

Given any other path γ' in $\overline{\Omega}$ from p to q, we have $\gamma' \in \overline{[\gamma]}$ if and only if there is a lift $\hat{\gamma}'$ of γ' satisfying $\hat{\gamma}'(0) = \hat{p}$ and $\hat{\gamma}'(1) = \hat{q}$.

Existence

We first establish the existence of a shortest path in $\overline{[\gamma]}$ via a construction in which we repeatedly replace subpaths of γ by straight line segments, when doing so does not change the homotopy class.

Let $\langle L_m \rangle_{m=1}^{\infty}$ be a countable dense family of distinct straight lines in $\mathbb{C}$; that is, for any pair of points $z_1, z_2 \in \mathbb{C}$ and any $\varepsilon > 0$, there is some m such that z_1 and z_2 are both within ε of the line L_m. For each m, the set $\Omega \cap L_m$ consists of at most countably many components (possibly none), each of which has countably many lifts to open arcs A in the covering space $\mathbb{D}$ such that $\overline{A}$ is an arc with endpoints on the boundary $\partial\mathbb{D}$. Let $\langle A_i \rangle_{i=1}^{\infty}$ enumerate the collection of all such lifts of components of $\Omega \cap L_m$, $m = 1, 2, 3, \ldots$.

We construct a sequence of paths γ_i, $i \geq 1$, by recursion. To begin, let $\gamma_1 = \gamma$. Having defined γ_i and its lift $\hat{\gamma}_i$, we define γ_{i+1} as follows:

- If $\hat{\gamma}_i^{-1}(A_i)$ has cardinality ≤ 1, then put $\gamma_{i+1} = \gamma_i$. Otherwise, let s_1 and s_2 be the smallest and largest (respectively) $s \in [0,1]$ such that $\hat{\gamma}_i(s) \in A_i$. Let $\hat{\gamma}_{i+1}$ be the path in $\mathbb{D}$ defined by $\hat{\gamma}_{i+1}(s) = \hat{\gamma}_i(s)$ for $s \notin [s_1, s_2]$, and $\hat{\gamma}_{i+1}{\restriction}_{[s_1,s_2]}$ parameterizes the subarc of A_i with endpoints $\hat{\gamma}_i(s_1)$ and $\hat{\gamma}_i(s_2)$ (or $\hat{\gamma}_{i+1}{\restriction}_{[s_1,s_2]}$ is constantly equal to w if $\hat{\gamma}_i(s_1) = \hat{\gamma}_i(s_2) = w$). Let $\gamma_{i+1} = \Phi \circ \hat{\gamma}_{i+1}$.

Claim 11.1. For all $\varepsilon > 0$, there exists a number N such that for all $i \geq 1$, there is no collection of more than N disjoint subintervals of $[0,1]$ whose images under γ_i have diameters $\geq \varepsilon$.

Proof of Claim 11.1. Fix $\varepsilon > 0$.

By construction, for any $x, t \in [0,1]$ and $\mu \in (0,1]$, the number of components C of the sets $\gamma_i^{-1}(S_j^{x,t,\mu})$ $(j \in \mathbb{Z})$ with $\mathrm{diam}(\mathrm{proj}_t^{\perp}(\gamma_i(C))) \geq \varepsilon$ is non-increasing with respect to i.

On the other hand, one can find K large enough so that if $t_l = \frac{l}{K}$ for $l = 0, 1, \ldots, K$, then for any connected set $P \subset \mathbb{C}$ of diameter $\geq \varepsilon$, there is an $l \in \{0, 1, \ldots, K\}$ and $j \in \mathbb{Z}$ such that $\mathrm{proj}_{t_l}^{\perp}(P \cap S_j^{0,t_l,\frac{\varepsilon}{3}})$ has diameter $\frac{\varepsilon}{3}$. It follows that if the claim were false for ε, there would be at least one $l \in \{0, 1, \ldots, K\}$ for which the number of components C of the sets $\gamma_i^{-1}(S_j^{0,t_l,\frac{\varepsilon}{3}})$ $(j \in \mathbb{Z})$ with $\mathrm{proj}_{t_l}^{\perp}(\gamma_i(C)) = \varepsilon$ would increase as i increases, contradicting the above observation. $\square$(Claim 11.1)

All of the paths γ_i are contained in the disk Δ. Therefore, by Theorem 9, there is a subsequence of $\langle \tilde{\gamma}_i \rangle_{i=1}^{\infty}$ (where $\tilde{\gamma}_i$ is the parameterization of γ_i by path length len) which converges to some path $\gamma_0 \in \overline{[\gamma]}$.

The path γ_0 lifts to a path $\hat{\gamma}_0 : [0,1] \to \overline{\mathbb{D}}$ with $\hat{\gamma}_0(0) = \hat{p}$, $\hat{\gamma}_0(1) = \hat{q}$.

Claim 11.2. For all $j > i \geq 1$, $\hat{\gamma}_j^{-1}(\overline{A_i}) \subset [0,1]$ is connected (possibly empty).

Proof of Claim 11.2. Since the lines in the family $\langle L_m \rangle_{m=1}^{\infty}$ are all distinct, it follows that any two arcs $\overline{A_i}$, $\overline{A_j}$ $(i \neq j)$ can meet in at most one point in $\overline{\mathbb{D}}$. Moreover, if A_i and A_j meet in $\mathbb{D}$, they cross transversally at their point of intersection, so that $\overline{A_i}$ separates the endpoints of $\overline{A_j}$ in $\overline{\mathbb{D}}$, and vice versa.

Fix an i. Clearly the claim holds for $j = i+1$ by construction of $\hat{\gamma}_{i+1}$.

Assume now that the claim holds for some $j > i$, and consider $\hat{\gamma}_{j+1}$. If $\hat{\gamma}_j^{-1}(\overline{A_i})$ is empty, then $\overline{A_i}$ does not separate $\hat{p}$ and $\hat{q}$ in $\overline{\mathbb{D}}$, and so since A_i and A_j are either disjoint or cross transversally, it is clear from the construction that $\hat{\gamma}_{j+1}^{-1}(\overline{A_i}) = \emptyset$.

Otherwise, $\hat{\gamma}_j^{-1}(A_i) = [t_1, t_2]$ for some $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$ (we allow $t_1 = t_2$ here, in which case $[t_1, t_2]$ is a single point $\{t_1\}$). We may assume that $\hat{\gamma}_j^{-1}(\overline{A_j})$ has at least two points, for otherwise $\hat{\gamma}_{j+1} = \hat{\gamma}_j$, and the claim holds trivially. Let s_1 and s_2 be the smallest and largest (respectively) $s \in [0,1]$ such that $\hat{\gamma}_j(s) \in \overline{A_j}$.

Then $s_1 < s_2$ by assumption. Since $\overline{A_i} \cap \overline{A_j}$ consists of at most one point, we cannot have both $s_1 \in [t_1, t_2]$ and $s_2 \in [t_1, t_2]$. This leaves five cases, enumerated below. In each case, the conclusion follows from the definition of $\hat{\gamma}_{j+1}$.

(i) If $s_1 < t_1$ and $s_2 > t_2$, then $\hat{\gamma}_{j+1}^{-1}(\overline{A_i})$ consists of a single point in the case that the subarc of $\overline{A_j}$ joining $\hat{\gamma}_j(s_1)$ and $\hat{\gamma}_j(s_2)$ meets $\overline{A_i}$, and is empty otherwise;

(ii) If $s_2 < t_1$, then $\hat{\gamma}_{j+1}^{-1}(\overline{A_i}) = [t_1, t_2]$, since in this case $\hat{\gamma}_j(s_1)$ and $\hat{\gamma}_j(s_2)$ are both on the same side of $\overline{A_i}$ in $\overline{\mathbb{D}}$, and so the subarc of $\overline{A_j}$ joining these two points cannot meet $\overline{A_i}$;

(iii) If $s_2 \in [t_1, t_2]$, then $\hat{\gamma}_{j+1}^{-1}(\overline{A_i}) = [s_2, t_2]$, since in this case $\hat{\gamma}_j(s_2) \in \overline{A_i}$ and so the subarc of $\overline{A_j}$ joining $\hat{\gamma}_j(s_1)$ and $\hat{\gamma}_j(s_2)$ is otherwise disjoint from $\overline{A_i}$;

(iv) If $s_1 \in [t_1, t_2]$, then $\hat{\gamma}_{j+1}^{-1}(\overline{A_i}) = [t_1, s_1]$, as in case (iii); and

(v) If $s_1 > t_2$, then $\hat{\gamma}_{j+1}^{-1}(\overline{A_i}) = [t_1, t_2]$, as in case (ii).

In any case, the set $\hat{\gamma}_{j+1}^{-1}(\overline{A_i})$ is connected, so the claim holds by induction. $\square$(Claim 11.2)

Claim 11.3. Let L be a straight line in $\mathbb{C}$, and let B be an arc in $\overline{\mathbb{D}}$ such that $\Phi(B) \subset L$. Then $\hat{\gamma}_0^{-1}(B) \subset [0, 1]$ is connected (possibly empty).

Proof of Claim 11.3. Suppose not, and let $s_1, s_2 \in [0, 1]$ be such that $s_1 < s_2$, $\hat{\gamma}_0(s_1) \in B$, $\hat{\gamma}_0(s_2) \in B$, and $\hat{\gamma}_0((s_1, s_2)) \cap B = \emptyset$.

Assume that $\hat{\gamma}_0(s_1) \neq \hat{\gamma}_0(s_2)$. Let C be a subarc of B between $\hat{\gamma}_0(s_1)$ and $\hat{\gamma}_0(s_2)$ with $\hat{\gamma}_0(s_1), \hat{\gamma}_0(s_2) \notin C$. Since $\hat{\gamma}_0([s_1, s_2])$ misses C, there is some neighborhood U of C in $\overline{\mathbb{D}}$ such that $\hat{\gamma}_0([s_1, s_2]) \cap U = \emptyset$.

Since the family $\langle L_m \rangle_{m=1}^{\infty}$ is dense, we can find lines in this family which are arbitrarily close to L, on either side of L, which do not meet L within the large disk Δ. Therefore, there are a lifts A_{i_1}, A_{i_2} which run close to C in U, one on each side of C in U (if $C \subset \partial\mathbb{D}$, we need only one lift) such that $\overline{A_{i_1}} \cup \overline{A_{i_2}} \cup U$ separate between $\hat{\gamma}_0(s_1)$ and $\hat{\gamma}_0(s_2)$ in $\overline{\mathbb{D}}$. It is straightforward to see that $\hat{\gamma}_0([s_1, s_2])$ must essentially cross one of $\overline{A_{i_1}}$ or $\overline{A_{i_2}}$, say $\overline{A_{i_1}}$, two times – once on either side of $\overline{A_{i_1}} \cap U$.

Since a subsequence of $\langle \hat{\gamma}_j \rangle_{j=1}^{\infty}$ converges uniformly to $\hat{\gamma}_0$, it follows that for large enough $j > i_1$ in this subsequence, the set $\hat{\gamma}_j^{-1}(\overline{A_{i_1}})$ is disconnected. But this contradicts Claim 11.2.

If $\hat{\gamma}_0(s_1) = \hat{\gamma}_0(s_2)$, then one can similarly find a lift A_i which is essentially crossed twice by $\hat{\gamma}_0([s_1, s_2])$, thus arriving again at a contradiction with Claim 11.2. $\square$(Claim 11.3)

Claim 11.4. γ_0 is a shortest path in $\overline{[\gamma]}$.

Proof of Claim 11.4. Let $s_1, s_2 \in [0, 1]$ with $s_1 < s_2$, and define the path γ_0' by $\gamma_0'(s) = \gamma_0(s)$ for $s \notin [s_1, s_2]$, and $\gamma_0' \restriction_{[s_1, s_2]}$ parameterizes the straight line segment $\overline{\gamma_0(s_1)\gamma_0(s_2)}$. Assume $\gamma_0' \in \overline{[\gamma]}$. Then there is a lift $\hat{\gamma}_0'$ of γ_0' such that $\hat{\gamma}_0' \restriction_{[0, s_1]} = \hat{\gamma}_0 \restriction_{[0, s_1]}$ and $\hat{\gamma}_0' \restriction_{[s_2, 1]} = \hat{\gamma}_0 \restriction_{[s_2, 1]}$.

Let $B = \hat{\gamma}_0'([s_1, s_2])$. Then B is an arc in $\overline{\mathbb{D}}$ such that $\Phi(B)$ is the straight line segment $\overline{\gamma_0(s_1)\gamma_0(s_2)}$. Since $\hat{\gamma}_0(s_1)$ and $\hat{\gamma}_0(s_2)$ are the endpoints of B, it follows from Claim 11.3 that $\hat{\gamma}_0([s_1, s_2]) = B$. Thus $\hat{\gamma}_0 = \hat{\gamma}_0'$ (up to reparameterization).

Therefore γ_0 is a shortest path in $\overline{[\gamma]}$. $\square$(Claim 11.4)

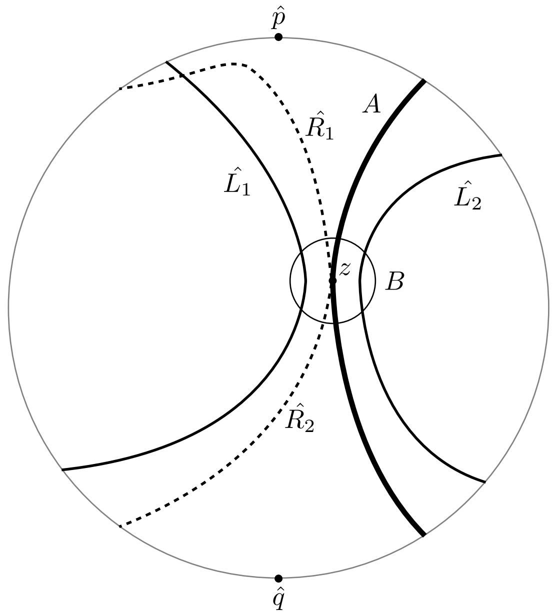

FIGURE 2. The situation described in Claim 11.5. In the scenario depicted here, the component of R_1 containing $\Phi(z)$ meets L_1 (in $\mathbb{C} \setminus \Delta$), while the component of R_2 containing $\Phi(z)$ does not (i.e. R_2 runs into the boundary $\partial\Omega$ before intersecting L_1).

Uniqueness

Let λ_1 and λ_2 be two shortest paths in $\overline{[\gamma]}$. Let $\hat{\lambda}_1$ and $\hat{\lambda}_2$ be lifts of λ_1 and λ_2 with $\hat{\lambda}_1(0) = \hat{\lambda}_2(0) = \hat{p}$ and $\hat{\lambda}_1(1) = \hat{\lambda}_2(1) = \hat{q}$.

Let A be a component of $\hat{\lambda}_1([0,1]) \setminus \partial\mathbb{D}$, so that $\Phi(A)$ is a straight line segment in Ω. We will show that $A \subset \hat{\lambda}_2([0,1])$.

Let $z \in A$, and fix $\varepsilon > 0$. Let $B \subset \mathbb{D}$ be a set containing z of diameter $< \varepsilon$ such that $\Phi(B)$ is a disk in Ω centered at $\Phi(z)$, and let B be small enough so that Φ is one-to-one on B. Then $\hat{\lambda}([0,1]) \cap B \subset A$, since otherwise λ_1 could be shortened by following a straight line segment within the disk $\Phi(B)$.

Let L_1 and L_2 be straight lines in $\mathbb{C}$ which are parallel to $\Phi(A)$, on either side of $\Phi(A)$, which both meet $\Phi(B)$. For $i = 1, 2$ let $\hat{L}_i$ be the closure in $\overline{\mathbb{D}}$ of the lift of the component of $L_i \cap \Omega$ which meets $\Phi(B)$, such that $\hat{L}_i$ meets B.

Claim 11.5. For $i = 1, 2$, $\hat{L}_i \cap \hat{\lambda}_1([0,1]) = \emptyset$.

Proof of Claim 11.5. Let R_1 and R_2 be two rays in $\mathbb{C}$ emanating from the point $\Phi(z)$, meeting at an obtuse angle there, and each intersecting the line L_1 at points r_1, r_2, respectively, such that r_1 and r_2 lie outside the large disk Δ which contains the path $\gamma([0,1])$. Thus R_1 and R_2 are each nearly parallel to L_1, and point in nearly opposite directions.

For $i = 1, 2$, let $\hat{R}_i \subset \mathbb{D}$ be the lift of the component of $R_i \cap \Omega$ containing $\Phi(z)$, such that $\hat{R}_i$ contains z. Then $\overline{\hat{R}_1} \cup \overline{\hat{R}_2}$ is an arc in $\overline{\mathbb{D}}$ with endpoints in $\partial\mathbb{D}$

(see Figure 2). Thus $\overline{\hat{R}_1} \cup \overline{\hat{R}_2}$ separates $\overline{\mathbb{D}}$ into two components, call them C_1 and C_2. Since λ_1 is a shortest path, we must have that $\hat{\lambda}_1([0,1]) \cap (\overline{\hat{R}_1} \cup \overline{\hat{R}_2}) = \{z\}$, for otherwise λ_1 could be shortened by following R_1 or R_2. Moreover, since the path $\hat{\lambda}_1$ does not cross $\hat{R}_1 \cup \hat{R}_2$ transversally at z, we have that $\hat{\lambda}_1([0,1]) \setminus \{z\}$ is contained entirely in one of C_1 or C_2, say C_2. In Figure 2, C_1 is the region on the left side of $\hat{R}_1 \cup \hat{R}_2$, and C_2 is the region on the right.

So $\hat{\lambda}_1([0,1]) \cap C_1 = \emptyset$. But clearly we also have $\hat{\lambda}_1([0,1]) \cap \Phi^{-1}(\Omega \setminus \Delta) = \emptyset$. Thus, since $\hat{L}_1 \subseteq C_1 \cup \Phi^{-1}(\Omega \setminus \Delta)$, it follows that $\hat{L}_1 \cap \hat{\lambda}_1([0,1]) = \emptyset$. Similarly, it can be seen that $\hat{L}_2 \cap \hat{\lambda}_1([0,1]) = \emptyset$. □(Claim 11.5)

It follows from Claim 11.5 that neither $\overline{\hat{L}_1}$ nor $\overline{\hat{L}_2}$ separates $\hat{p}$ from $\hat{q}$ in $\overline{\mathbb{D}}$. Thus there is a component K of $\mathbb{D} \setminus (\hat{L}_1 \cup \hat{L}_2)$ such that $\hat{p}, \hat{q} \in \overline{K}$. Then $\hat{\lambda}_2([0,1]) \subset \overline{K}$. But B separates $\overline{K}$, and since $(\hat{\lambda}_1([0,1]) \setminus A) \cap B = \emptyset$, we have that B separates $\hat{p}$ from $\hat{q}$ in $\overline{K}$. This implies $\hat{\lambda}_2([0,1]) \cap B \neq \emptyset$. Since ε was arbitrary, it follows that $z \in \hat{\lambda}_2([0,1])$.

Thus, $\hat{\lambda}_1([0,1]) \cap \mathbb{D} = \hat{\lambda}_2([0,1]) \cap \mathbb{D}$. It is straightforward to see that this implies that $\hat{\lambda}_1 = \hat{\lambda}_2$ up to reparameterization. □

References

1. Leonard M. Blumenthal and Karl Menger, *Studies in geometry*, W. H. Freeman and Co., San Francisco, Calif., 1970.
2. Richard D. Bourgin and Peter L. Renz, *Shortest paths in simply connected regions in* $\mathbf{R}^2$, Adv. Math. **76** (1989), no. 2, 260–295.
3. Herbert Busemann, *The geometry of geodesics*, Academic Press Inc., New York, N. Y., 1955.
4. J. W. Cannon, G. R. Conner, and Andreas Zastrow, *One-dimensional sets and planar sets are aspherical*, Topology Appl. **120** (2002), no. 1-2, 23–45, In memory of T. Benny Rushing. MR 1895481 (2003g:57028)
5. John B. Conway, *Functions of one complex variable. II*, Graduate Texts in Mathematics, vol. 159, Springer-Verlag, New York, 1995.
6. Gerald B. Folland, *Real analysis: modern techniques and their applications*, second ed., Pure and Applied Mathematics (New York), John Wiley & Sons Inc., New York, 1999.
7. William S. Massey, *Algebraic topology: An introduction*, Harcourt, Brace & World, Inc., New York, 1967.
8. Karl Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. **100** (1928), no. 1, 75–163.
9. Sam B. Nadler, Jr., *Continuum theory: an introduction*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 158, Marcel Dekker Inc., New York, 1992.